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Events and Observables in Axiomatic Quantum Mechanics

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Abstract

It is shown that in fairly general circumstances the event and observable frameworks for axiomatic quantum mechanics are equivalent.

Introduction

In the quantum logic approach to axiomatic quantum mechanics (Birkhoff and von Neumann, 1936; Mackey, 1963) the basic mathematical structure is the partially ordered set of events, which are usually required to form a σ-orthomodular poset (Mackey, 1963; Piron, 1963; Jauch, 1968; Pool, 1968). In this framework a set of events is said to be compatible (or simultaneously measurable) if it is contained in a Boolean sub-σ-algebra of the poset; thus it is natural to think of such subalgebras as the mathematical objects corresponding to measurements.

In this paper we generalize this context by studying what we call a system of measurements: a collection of Boolean σ-algebras that overlap in such a way that operations (complements, countable joins) are consistently defined on the intersections. We characterize those systems of measurements that arise from quantum logics (our terminology for σ-orthomodular posets). Related results appear in Finch (1969) and Maczynski, (1970).

An alternative approach to axiomatic quantum mechanics takes observables as the fundamental objects (Jordan et al., 1934; Segal, 1947; Deliyannis, 1969). Observables also arise in the event framework as homomorphisms from $B(\mathbb{R})$ (the Borel sets of the reals $\mathbb{R}$) into the quantum logic. In either framework one can define an action of suitable real-valued functions on the set of observables.

We define a system of observables as a set together with an action of $BF(\mathbb{R})$ (the real-valued Borel functions on $\mathbb{R}$) on that set. We say a system of observables is standard if sums of simultaneously measurable observables are well defined. [These systems were first defined and studied by Deliyannis (1969).] With each system of measurements there is associated a system of observables; we show that the systems of observables that arise in this way are precisely the standard ones.

Our main results concern the relationship between systems of measurements and systems of observables. To investigate equivalence of event and observable frameworks we define a functor Q from the category $\mathscr{X}$ of systems of measurements to the category $\mathscr{Q}$ of systems of observables. This functor is not an equivalence but on certain subcategories of $\mathscr{X}$ becomes one. These subcategories are characterized as those possessing the "induced map property". We give sufficient conditions for the induced map property in terms of separability and compatibility requirements, and thus can define in easily verifiable terms two subcategories on which Q is an equivalence. One of these subcategories contains all systems of measurements derived from quantum logics, leading to the corollary that a quantum logic is determined up to isomorphism by its associated system of observables. Finally, we show that if we modify the category $\mathscr{X}$ (by slightly weakening the notion of homomorphism), then the modified category $\tilde{\mathscr{X}}$ is equivalent to $\mathscr{Q}$.

2. *Systems of Measurements*

Let $\mathscr{M}$ be a collection (not necessarily disjoint) of Boolean σ-algebras. [By a Boolean σ-algebra we mean a σ-complete distributive complemented lattice; see Halmos (1963) for relevant definitions and background.] We say $\mathscr{M}$ is *consistent* if every $M \in \mathscr{M}$ has the same least element 0, and if, for all M_1 and M_2 in $\mathscr{M}$, the operations of M_1 and M_2 (complements and countable joins) agree on $M_1 \cap M_2$. A *system of measurements* is a pair $(X, \mathscr{M})$ where $\mathscr{M}$ is a consistent collection of Boolean σ-algebras, and X equals the union of the sets $M \in \mathscr{M}$. If A is a subset of X, we say A is *compatible* if $A \subseteq M$ for some $M \in \mathscr{M}$. If A is a countable compatible subset of X, then $\vee A$ denotes the join of A (as calculated in any $M \in \mathscr{M}$ containing A). For $x \in X$, x' denotes the complement of x (found in any $M \in \mathscr{M}$ containing x). When no confusion will result we will write X in place of $(X, \mathscr{M})$.

One example of such a system is $(X, \{M\})$, where X is any set and M is a collection of subsets of X closed under set complementation and countable union, i.e., (X, M) is a measurable space. For a second example, we begin by reviewing some facts about orthomodular posets. Let $(L, \lesssim)$ be a partially ordered set (poset) with least element 0 and greatest element 1. A map $': L \to L$ is an orthocomplementation if $'$ satisfies $(x')' = x$, $x \lesssim y$ implies $y' \lesssim x'$, and $x \vee x' = 1$. An orthocomplemented poset $(L, \lesssim, ')$ is an orthomodular poset if $x \lesssim y'$ implies $x \vee y$ exists, and if $x \lesssim y$ implies $y = x \vee (y \wedge x')$. An orthomodular poset $(L, \lesssim, ')$ is a *quantum logic* if whenever a sequence $\{x_i\}$ in L satisfies $x_i \lesssim x_j'$ for $i \neq j$, then $\vee x_i$ exists in L.

Let $(L, \lesssim, ')$ be a quantum logic. A subset $A \subseteq L$ is a Boolean sub-σ-algebra of L if (i) for every countable subset $E \subseteq A$, $\vee E$ exists and is in A; (ii) if $a \in A$ then $a' \in A$; and (iii) $(A, \lesssim, ')$ is a Boolean algebra. The collection of Boolean sub-σ-algebras of L will be denoted $\mathscr{M}(L)$. It is easily verified that $\mathscr{M}(L)$ is consistent, and so $(L, \mathscr{M}(L))$ is a system of measurements.

If $(X, \mathscr{M})$ and $(X', \mathscr{M}')$ are systems of measurements, then a map $h: X \to X'$ is a *homomorphism* if for each $M \in \mathscr{M}$ there exists $M' \in \mathscr{M}'$ such that $h|_M$ is a σ-homomorphism of M into M'. (Recall that a σ-homomorphism of Boolean σ-algebras is a map preserving complements and countable joins.) We say h is an *isomorphism* if h is bijective and both h and h^{-1} are homomorphisms. If L_1 and L_2 are quantum logics we say $h: L_1 \to L_2$ is a σ-homomorphism if (i) $h(0) = 0$ and $h(1) = 1$, and (ii) if $\{x_i\}$ is a sequence in L_1 such that $x_i \lesssim x_j'$ for $i \neq j$, then $h(x_i) \lesssim h(x_j)'$ for $i \neq j$, and $h(\vee x_i) = \vee h(x_i)$. It is not difficult to show $h: L_1 \to L_2$ is a σ-homomorphism iff $h: (L_1, \mathscr{M}(L_1)) \to (L_2, \mathscr{M}(L_2))$ is a homomorphism (cf. Shultz, 1972, p. 38).

We will now proceed to characterize those systems $(X, \mathscr{M})$ for which there exists a quantum logic L such that $(X, \mathscr{M}) \simeq (L, \mathscr{M}(L))$. The natural candidate for a partial ordering on X is the relation $x \lesssim y$ if $x \vee y = y$ in some $M \in \mathscr{M}$ containing x and y. This is easily seen to be reflexive and antisymmetric, but it is not necessarily transitive. We say $(X, \mathscr{M})$ is *transitive* if the relation $\lesssim$ defined above is transitive.

For an arbitrary system of measurements $(X, \mathscr{M})$ we say $x, y \in X$ are orthogonal if $x \lesssim y'$. We say $(X, \mathscr{M})$ satisfies Property $\mathscr{O}$ if every countable pairwise orthogonal subset of X is compatible.

The following result appears (in slightly weaker form) in Finch (1969, Theorems 1.2, 3.1).

> *Lemma 2.1.* (i) If $(X, \mathscr{M})$ is transitive and satisfies Property $\mathscr{O}$, then $(X, \lesssim, ')$ is a quantum logic; (ii) If L is a quantum logic, then $(L, \mathscr{M}(L))$ is transitive and satisfies Property $\mathscr{O}$.

Proof. Assume $(X, \mathscr{M})$ is transitive and satisfies Property $\mathscr{O}$. Then $(X, \lesssim)$ is a partially ordered set, and the map $'$ satisfies (i) $(x')' = x$ and (ii) $x \lesssim y$ implies $y' \lesssim x'$. The least element of X is 0 and the greatest is $1 = 0'$.

Now let $\{x_i\}$ be any orthogonal sequence in X. Then by Property $\mathscr{O}$ there exists M in $\mathscr{M}$ with $\{x_i\} \subseteq M$. Let $x = \vee x_i$ (join in M); we claim x is the least upper bound of $\{x_i\}$ in $(X, \lesssim)$. Suppose $x_i \lesssim y \in X$ for all i; then $x_i \lesssim (y')'$ so $\{y'\} \cup \{x_1, x_2, \ldots\}$ is a countable orthogonal subset of X. By Property $\mathscr{O}$ there exists M' in $\mathscr{M}$ containing $\{y'\} \cup \{x_1, x_2, \ldots\}$ and thus also containing $y = (y')'$. By the consistency of $\mathscr{M}$, $x = \vee x_i$ (join in M'); since $x_i \lesssim y$ for all i, then $x \lesssim y$. Thus we have shown that x is the least upper bound of $\{x_i\}$ in $(X, \lesssim)$.

Thus the least upper bound of every orthogonal sequence exists (and agrees with the join calculated in any $M \in \mathscr{M}$ containing that sequence). In particular, for each $x \in X$ the least upper bound of x and x' is 1, proving that $(X, \lesssim, ')$ is an orthocomplemented poset. Finally, if $x \lesssim y$, then there exists M in $\mathscr{M}$ containing x and y; therefore, $y = x \vee (y' \vee x)' = x \vee (y \wedge x')$

as calculated either in M or in $(X, \lesssim, ')$. Thus the orthomodular identity holds in $(X,\lesssim, ')$, completing the proof that $(X,\lesssim, ')$ is a quantum logic.

Now let L be any quantum logic. If $x \lesssim y$ (where $\lesssim$ is the ordering of the poset L), then x and y are compatible, so the ordering induced from $(L,\mathscr{M}(L))$ agrees with the original ordering of L. It follows that $(L,\mathscr{M}(L))$ is transitive. Let $A \subseteq L$ be a countable orthogonal subset and let $x = \bigvee A$. Then if $\tilde{A} = A \cup \{x'\}$, $\tilde{A}$ is a countable orthogonal subset of L and $\bigvee\tilde{A} = 1$. By a result of Ramsay (1966, Lemma 2) (proven for $\tilde{A}$ finite but valid with the same proof for $\tilde{A}$ countably infinite), $\tilde{A}$ is contained in a Boolean sub-σ-algebra of L. Thus A is compatible, showing that $(L,\mathscr{M}(L))$ satisfies Property $\mathscr{O}$. □

Let $(X,\mathscr{M})$ be a transitive system of measurements satisfying Property $\mathscr{O}$. Then $L = (X,\lesssim, ')$ is a quantum logic. However, it is not necessarily true that $(X,\mathscr{M}) \cong (L,\mathscr{M}(L))$; in fact, there may not exist any quantum logic L with $(X,\mathscr{M}) \cong (L, \mathscr{M}(L))$. The problem is that if L is a quantum logic, then, in general, there will exist collections $\mathscr{M} \subseteq \mathscr{M}(L)$ that are rich enough to determine the order structure of L but not rich enough to determine the correct compatibility structure. To achieve the result that we are after we therefore have to specify an additional property. We say $(X,\mathscr{M})$ satisfies the *finite compatibility* property if whenever A is a subset of X such that every finite subset of A is compatible, then A is compatible. We remark that the properties of transitivity, Property $\mathscr{O}$, and finite compatibility are independent (Shultz, 1972, Proposition 4.15). Here we will give an example where the former two properties hold but the last fails. Let B be a Boolean σ-algebra that is not atomic. Let $\mathscr{M}_a(B)$ be the collection of atomic Boolean sub-σ-algebras of B. Then $(B, \mathscr{M}_a(B))$ is transitive and satisfies Property $\mathscr{O}$, but does not have the finite compatibility property.

We can now state the main result of this section.

> *Theorem 2.2.* Let $(X,\mathscr{M})$ be a system of measurements. Then there exists a quantum logic L such that $(L,\mathscr{M}(L))$ is isomorphic to $(X,\mathscr{M})$ iff $(X,\mathscr{M})$ is transitive, satisfies Property $\mathscr{O}$, and has the finite compatibility property. In this case $(X,\mathscr{M})$ is isomorphic to $(L,\mathscr{M}(L))$ where $L = (X,\lesssim, ')$.

Proof. We first show that the conditions mentioned are necessary. Suppose L is a quantum logic and $(X,\mathscr{M}) \cong (L,\mathscr{M}(L))$. By Lemma 2.1 $(L,\mathscr{M}(L))$ [and therefore $(X,\mathscr{M})$] is transitive and satisfies Property $\mathscr{O}$. Let A be a subset of L such that every finite subset of A is compatible. For each finite subset E of A let $[E]$ denote the smallest Boolean sub-σ-algebra of L containing E. Let $B = \cup\{[E] \mid E \text{ is a finite subset of } A\}$; we claim B is compatible. If x,y are in B, say, $x \in [E_1]$ and $y \in [E_2]$, then x and y are in $[E_1 \cup E_2]$, so that every pair of elements in B is compatible. Also, $x \vee y$ and x' are in $[E_1 \cup E_2] \subseteq B$. It follows that B is a Boolean subalgebra of L (Varadarajan, 1962, Proposition 3.9) and so B is contained in a Boolean sub-σ-algebra of L (Ramsay, 1966, Lemma 11). Since $A \subseteq B$, we have shown A is compatible, and thus $(X,\mathscr{M}) \cong (L,\mathscr{M}(L))$ satisfies the finite compatibility property.

Conversely, suppose $(X,\mathscr{M})$ is a system that is transitive, satisfies Property $\mathcal{O}$, and has the finite compatibility property. Let L be the quantum logic $(X,\lesssim,')$; we will show $(X,\mathscr{M})$ is isomorphic to $(L,\mathscr{M}(L))$.

We will first show that $\mathscr{M} \subseteq \mathscr{M}(L)$. Suppose $M \in \mathscr{M}$; we must show M is a Boolean sub-σ-algebra of L. By assumption each $M \in \mathscr{M}$ is a Boolean σ-algebra. By definition of the map $'$: $X \to X$, if $x \in M$, then $x' \in M$. There remains to show that countable subsets of M have their least upper bound in M. Let $\{x_i\}$ be a sequence in M. Define $y_1 = x_1$, and

$$y_i = x_i \wedge (\bigvee_{j<i} x_j)'$$

for $i > 1$, with meets and joins in M. Then $\{y_i\}$ is an orthogonal sequence in M; let $y = \vee y_i$ (join in M). We will show that y is the least upper bound in X of $\{x_i\}$. Since

$$\bigvee_1^n x_i = \bigvee_1^n y_i$$

(joins in M) for all n, then

$$y \geqslant \bigvee_1^n x_i$$

for all n, and thus $y \geqslant x_i$ for all i. On the other hand, if $z \in X$ and $z \geqslant x_i$ for all i, then $z \geqslant x_i \geqslant y_i$ all i. As shown in the proof of Lemma 2.1, $y = \vee y_i$ is the least upper bound of $\{y_i\}$ in X, so $z \geqslant y$. Thus y is the least upper bound of $\{x_i\}$ in X, which completes the proof that $\mathscr{M} \subseteq \mathscr{M}(L)$.

Let ψ be the identity map on X. Since $\mathscr{M} \subseteq \mathscr{M}(L)$ it follows that ψ: $(X,\mathscr{M}) \to (X = L,\mathscr{M}(L))$ is a homomorphism. We will be finished if we can show that every $B \in \mathscr{M}(L)$ is a Boolean sub-σ-algebra of some $M \in \mathscr{M}$ [and so ψ^{-1} is a homomorphism, proving $(X,\mathscr{M}) \cong (L,\mathscr{M}(L))$]. We now use the finite compatibility property of $(X,\mathscr{M})$. Let E be any finite subset of B, and let $[E]$ denote the Boolean sub-σ-algebra of B generated by E. Since E is finite, then $[E]$ will also be finite. Let $\{a_1, \ldots, a_n\}$ be the atoms (minimal nonzero elements) of $[E]$; each $e \in E$ can then be expressed as the least upper bound of some subset of $\{a_1, \ldots, a_n\}$. Since a_i and a_j are orthogonal for $i \neq j$, by Property $\mathcal{O}$ there exists M in $\mathscr{M}$ containing $\{a_1, \ldots, a_n\}$. Since, as shown above, M is a Boolean sub-σ-algebra of L, then $E \subseteq M$. Thus every finite subset E of B is compatible in $(X,\mathscr{M})$; by the finite compatibility property B is compatible, i.e., B is contained in some $M \in \mathscr{M}$. Since B and M are both Boolean sub-σ-algebras of L, then B is a Boolean sub-σ-algebra of M. This completes the proof that $(X,\mathscr{M})$ and $(L,\mathscr{M}(L))$ are isomorphic. □

3. *Systems of Observables*

Let $BF(\mathbb{R})$ be the set of Borel functions from $\mathbb{R}$ to $\mathbb{R}$. By an action of

$BF(\mathbb{R})$ on a set Q we mean a map $(f, q) \to f \cdot q$ of $BF(\mathbb{R}) \times Q$ into Q, satisfying

$$f \cdot (g \cdot q) = (f \circ g) \cdot q \text{ for all } f, g \in BF(\mathbb{R}), q \in Q \tag{3.1}$$

$$i_{\mathbb{R}} \cdot q = q \text{ for all } q \in Q, \text{ where } i_{\mathbb{R}}(\lambda) = \lambda \text{ for all } \lambda \in \mathbb{R} \tag{3.2}$$

A set Q with an action of $BF(\mathbb{R})$ we call a *system of observables.*

As an example, let L be any quantum logic. An *observable* of L is a σ-homomorphism from the Borel sets $B(\mathbb{R})$ into L. Let $Q(L)$ be the set of observables of L and for $f \in BF(\mathbb{R})$, $q \in Q(L)$ define $f \cdot q \in Q(L)$ by

$$(f \cdot q)(E) = q[f^{-1}(E)] \text{ for all } E \in B(\mathbb{R}) \tag{3.3}$$

Then $Q(L)$ with this action of $BF(\mathbb{R})$ is a system of observables.

We next discuss another example generalizing the one above. Let $(X, \mathscr{M})$ be any system of measurements. An *observable* of $(X, \mathscr{M})$ is a homomorphism from $(B(\mathbb{R}), \{B(\mathbb{R})\})$ into $(X, \mathscr{M})$. The set of observables of $(X, \mathscr{M})$ we denote by $Q(X)$. For $f \in BF(\mathbb{R})$ and $q \in Q(X)$ we define $f \cdot q \in Q(X)$ by (3.3); then $Q(X)$ becomes a system of observables.

Our purpose in this section is to investigate which systems of observables arise as a system $Q(X)$ associated with a system of measurements. If Q_1 and Q_2 are systems of observables, then $\varphi: Q_1 \to Q_2$ is a homomorphism if

$$\varphi(f \cdot q) = f \cdot \varphi(q) \text{ for all } f \in BF(\mathbb{R}), q \in Q_1 \tag{3.4}$$

and φ is an isomorphism if it is bijective in addition. Our goal is to determine for which systems of observables there exists an isomorphic system of the form $Q(X)$.

Let Q be any system of observables and suppose $\{q_i, i \in I\} \subseteq Q$ has the property that there exists $q \in Q$ and $\{f_i, i \in I\} \subseteq BF(\mathbb{R})$ such that $q_i = f_i \cdot q$ for all $i \in I$. Then physically we can measure the observables q_i simultaneously by measuring the value of q and then applying the functions $\{f_i\}$. We can also measure the "sum" of the observables $\{q_i\}$ by finding $(\Sigma f_i) \cdot q$. However, in general, this sum may depend upon the particular choice of q and $\{f_i\}$. We will say a system of observables is standard if such sums are well defined:

Definition. Let Q be a system of observables. We say Q is *standard* if

Whenever q_1 and q_2 are in Q, $\{f_i, i \in I\}$ and $\{q_i, i \in I\}$ are in $BF(\mathbb{R})$, I is at most countable, Σf_i and Σg_i converge everywhere, and $f_i \cdot q_1 = g_i \cdot q_2$ for all $i \in I$, then $(\Sigma f_i) \cdot q_1 = (\Sigma g_i) \cdot q_2$ (3.5)

[The axiom (3.5) is due to Deliyannis (1969).] Before characterizing the systems $Q(X)$ we need the following technical lemma.

Lemma 3.1. Let Q be a standard system of observables. For each $q \in Q$ let $M_q = \{\chi_E \cdot q \mid E \in B(\mathbb{R})\}$, and define $\vee$ and $'$ on M_q by

$$(\chi_E \cdot q)' = \chi_{\mathbb{R} - E} \cdot q; \quad \vee (\chi_{E_i} \cdot q) = \chi_{\cup E_i} \cdot q \tag{3.6}$$

Then each M_q is a Boolean σ-algebra, and if $\mathscr{M}(Q) = \{M_q \mid q \in Q\}$,

$$X(Q) = \bigcup_{q \in Q} M_q$$

then $(X(Q), \mathscr{M}(Q))$ is a system of measurements.

Proof. Note first that $'$ is well defined, since, if $\chi_E \cdot q_1 = \chi_F \cdot q_2$, then applying $\chi_\emptyset$ to both sides and using (3.1) yields $\chi_{R-E} \cdot q_1 = \chi_{R-F} \cdot q_2$. In order to establish that $\vee$ is well defined, observe first that

$$\text{If } f_i \cdot q_1 = g_i \cdot q_2 \text{ (for } i = 1, 2\text{), then } (f_1 f_2) \cdot q_1 = (g_1 g_2) \cdot q_2 \tag{3.7}$$

{This follows from (3.1), (3.5), and the identity $h_1 h_2 = \frac{1}{2}[(h_1 + h_2)^2 - h_1^2 - h_2^2]$.} Thus

$$\text{If } \chi_{E_i} \cdot q_1 = \chi_{F_i} \cdot q_2 (i = 1, 2, \ldots), \text{ then } \chi_{\cup E_i} \cdot q_1 = \chi_{\cup F_i} \cdot q_2 \tag{3.8}$$

$$[\{E_i\}, \{F_i\} \subseteq B(\mathbb{R}), q_1, q_2 \in Q]$$

follows from (3.5), (3.7), and the identity

$$\chi_{E_i} = \chi_{E_1} + \chi_{E_2}\chi_{R-E_1} + \chi_{E_3}\chi_{R-E_2}\chi_{R-E_1} + \cdots$$

(with pointwise convergence on the right). It follows that $\vee$ is well defined on M_q. It is now easily verified that each M_q is a Boolean σ-algebra. By (3.5) with $I = \emptyset$, $\chi_\emptyset \cdot q_1 = \chi_\emptyset \cdot q_2$ for all q_1, q_2 in Q so each M_q has the same zero. From this and (3.8) it follows that $\mathscr{M}(Q)$ is consistent and thus $(X(Q), \mathscr{M}(Q))$ is a system of measurements. □

Theorem 3.2. If Q is a system of observables, then there exists a system of measurements $(X, \mathscr{M})$ such that Q is isomorphic to $Q(X)$ iff Q is standard.

Proof. We show first that each system $Q(X)$ is standard. Assume $\{f_i\}$, $\{g_i\}$, q_1, q_2 satisfy the hypotheses of (3.5) with the index set I being non-empty. Define $\mathbf{f}: \mathbb{R} \to \mathbb{R}^I$ by $\mathbf{f}(\lambda) = (f_1(\lambda), f_2(\lambda), \ldots)$ and $\mathbf{g}$ similarly; note that $\mathbf{f}$ and $\mathbf{g}$ are Borel functions. Let $\pi_i: \mathbb{R}^I \to \mathbb{R}$ be the projection on the ith component; observe that $\mathbf{f}^{-1}(\pi_i^{-1}(E)) = f_i^{-1}(E)$ for all $E \in B(\mathbb{R})$ and similarly for $\mathbf{g}$. Thus for each $E \in B(\mathbb{R})$

$$\begin{aligned}(q_1 \circ \mathbf{f}^{-1})[\pi_i^{-1}(E)] &= q_1(f_i^{-1}(E)) \\ &= (f_i \cdot q_1)(E) \\ &= (g_i \cdot q_2)(E) = (q_2 \circ \mathbf{g}^{-1})[\pi_i^{-1}(E)]\end{aligned}$$

Since $\{\pi_i^{-1}(E) \mid i \in I, E \in B(\mathbb{R})\}$ generate $B(\mathbb{R}^I)$, it follows that $q_1 \circ \mathbf{f}^{-1}$ and $q_2 \circ \mathbf{g}^{-1}$ agree on $B(\mathbb{R}^I)$.

Now define $h: \mathbb{R}^I \to \mathbb{R}$ by $h(\lambda_1, \lambda_2, \ldots) = \Sigma\, \lambda_i$ if the sum converges, zero otherwise. Then h is a Borel function and $h \circ \mathbf{f} = \Sigma\, f_i$, $h \circ \mathbf{g} = \Sigma\, g_i$. Thus for $E \in B(\mathbb{R})$

$$\begin{aligned}(\Sigma\, f_i \cdot q_1)(E) &= (q_1 \circ \mathbf{f}^{-1} \circ h^{-1})(E) \\ &= (q_2 \circ \mathbf{g}^{-1} \circ h^{-1})(E) = (\Sigma\, g_i \cdot q_2)(E)\end{aligned}$$

which establishes (3.5) for $I \neq \emptyset$. If I is empty, we must show $0 \cdot q_1 = 0 \cdot q_2$ for all q_1, q_2 in $Q(X)$. This follows from $q_1(\emptyset) = q_2(\emptyset) = 0$ (the common zero of the algebras $M \in \mathscr{M}$), and the proof that $Q(X)$ is standard is completed.

Conversely, let Q_0 be a standard system of observables. Let X be the system of measurements $(X(Q_0), \mathscr{M}(Q_0))$ defined in Lemma 3.1; we will show $Q_0 \cong Q(X)$. For each $q \in Q_0$ define $\hat{q} \in Q(X)$ by $\hat{q}(E) = \chi_E \cdot q$ (for all $E \in B(\mathbb{R})$). Note that

$$\begin{aligned}(f \cdot q)\hat{}\,(E) = \chi_E \cdot (f \cdot q) &= (\chi_E \circ f) \cdot q \\ &= \chi_{f^{-1}(E)} \cdot q \\ &= \hat{q}(f^{-1}(E)) \\ &= (f \cdot \hat{q})(E)\end{aligned}$$

so $q \mapsto \hat{q}$ is a homomorphism of Q_0 onto $Q(X)$.

To see that the map $q \mapsto \hat{q}$ is one to one, suppose $\hat{q}_1 = \hat{q}_2$, so that $\chi_E \cdot q_1 = \chi_E \cdot q_2$ for all $E \in B(\mathbb{R})$. By (3.1) and (3.5), finite linear combinations of characteristic functions will agree on q_1 and q_2. The identity function $i_{\mathbb{R}}$ is (pointwise) the sum of a suitable sequence of such functions. Thus, by (3.5), $i_{\mathbb{R}} \cdot q_1 = i_{\mathbb{R}} \cdot q_2$ and so by (3.2) $q_1 = q_2$.

We now will show $q \mapsto \hat{q}$ is surjective. Suppose h: $B(\mathbb{R}) \to X$ is any observable of $Q(X)$ with h: $B(\mathbb{R}) \to M_q$ being a σ-homomorphism for some $q \in Q_0$. Since $\hat{q}$ is a σ-homomorphism of $B(\mathbb{R})$ onto M_q, there will exist $f \in BF(\mathbb{R})$ such that $h = \hat{q} \circ f^{-1}$ (Varadarajan, 1968, Theorem 1.4), and thus $h = f \cdot \hat{q} = (f \cdot q)\hat{}$. This completes the proof that $q \mapsto \hat{q}$ is an isomorphism of Q_0 onto $Q(X)$. □

4. *Equivalence of Event and Observable Framework*

To establish the equivalence of the two approaches discussed in Section 2 and 3 (events and observables) one would hope to show that the category $\mathscr{X}$ of systems of measurements and the category $\mathscr{Q}$ of standard systems of observables are equivalent. Recall that if $\mathscr{A}$, $\mathscr{B}$ are categories and T: $\mathscr{A} \to \mathscr{B}$ is a functor, then T is an *equivalence* if

for each $B \in \mathscr{B}$ there exists $A \in \mathscr{A}$ with $T(A) \cong B$, (4.1)

for each pair A_1, A_2 in $\mathscr{A}$, T: $\hom[A_1, A_2] \to \hom[T(A_1), T(A_2)]$ is bijective. (4.2)

If X_1 and X_2 are systems of measurements and ψ: $X_1 \to X_2$ is a homomorphism, then ψ induces the homomorphism ψ^*: $Q(X_1) \to Q(X_2)$ given by $\psi^*(q) = \psi \circ q$. If we define $Q(\psi) = \psi^*$, then $X \mapsto Q(X)$, $\psi \mapsto Q(\psi)$ is a functor from $\mathscr{X}$ to $\mathscr{Q}$. Ideally, the functor Q: $\mathscr{X} \to \mathscr{Q}$ would be an equivalence; unfortunately, this is not the case. By Theorem 3.2, condition (4.1) holds, but (4.2) fails. As an example, let B be a non-separable Boolean σ-algebra. (Recall B is separable iff it is countably generated.) Let $\mathscr{S}(B)$ be the collection of separable Boolean sub-σ-algebras of B. Since $B(\mathbb{R})$ is separable, then the range of every observable $q: B(\mathbb{R}) \to B$ is always separable; therefore, $Q(B, \{B\}) = Q(B, \mathcal{S}(B))$, but $(B, \{B\})$ and $(B, \mathcal{S}(B))$ are not isomorphic, so $Q: \mathcal{X} \to \mathcal{Q}$ is not an equivalence. [Note that if $T: \mathcal{A} \to \mathcal{B}$ is an equivalence, then by (4.2) $T(A_1) \cong T(A_2)$ implies $A_1 \cong A_2$.]

More modestly, we can hope that $Q: \mathcal{X}_0 \to \mathcal{Q}_0$ (with $\mathcal{X}_0 \subseteq \mathcal{X}$, $\mathcal{Q}_0 \subseteq \mathcal{Q}$) will be an equivalence for suitable subcategories $\mathcal{X}_0, \mathcal{Q}_0$ large enough to contain the main cases of interest. If (4.2) holds on $\mathcal{X}_0$, then we can choose $\mathcal{Q}_0$ to be the image of $\mathcal{X}_0$; thus our main concern is to investigate when $Q: \hom[X_1, X_2] \to \hom[Q(X_1), Q(X_2)]$ is bijective. We characterize the pairs (X_1, X_2) for which this occurs in Theorem 4.1, which depends upon the following definition.

Definition. Let X and X' be systems of measurements. The pair (X, X') is said to have the *induced map property* if whenever $\psi: X \to X'$ has the property that $\psi \circ q$ is in $Q(X')$ for every q in $Q(X)$, then ψ is a homomorphism from X to X'.

The following observation will be useful in the proof of Theorem 4.1. If M is any Boolean σ-algebra and $x \in M$, then one can easily construct a σ-homomorphism from $B(\mathbb{R})$ onto the subalgebra $\{0, x, x', 1\}$. Thus if $(X, \mathcal{M})$ is any system of measurements, then every $x \in X$ is in the image of a suitable observable $q \in Q(X)$.

Theorem 4.1. Let X and X' be systems of measurements. Then $Q: \hom[X, X'] \to \hom[Q(X), Q(X')]$ is bijective iff the pair (X, X') has the induced map property.

Proof. Assume $Q: \hom[X, X'] \to \hom[Q(X), Q(X')]$ is bijective. To verify the induced map property, suppose $\psi: X \to X'$ has the property that $\psi \circ q \in Q(X')$ for all $q \in Q(X)$. Then $q \to \psi \circ q$ is a homomorphism from $Q(X)$ to $Q(X')$, and so since Q is bijective there will exist $\varphi \in \hom[X, X']$ such that $Q(\varphi)$ equals the map $q \mapsto \psi \circ q$, i.e., $[Q(\varphi)](q) = \varphi \circ q = \psi \circ q$ for all q in $Q(X)$. Since every element of X is in the range of a suitable observable $q \in Q(X)$, it follows that $\psi = \varphi \in \hom[X, X']$, showing that (X, X') has the induced map property.

Conversely, assume (X, X') has the induced map property. We first show that $Q: \hom[X, X'] \to \hom[Q(X), Q(X')]$ is one to one. Suppose ψ_1 and ψ_2 are in $\hom[X, X']$ and $Q(\psi_1) = Q(\psi_2)$. Then, for all $q \in Q(X)$, $\psi_1 \circ q = \psi_2 \circ q$; as above, it follows that $\psi_1 = \psi_2$. (Note this part of the proof is valid for arbitrary X, X' in $\mathcal{X}$.)

To show Q is surjective, suppose φ is in $\hom[Q(X), Q(X')]$. Define $\psi: X \to X'$ by $\psi(q(E)) = [\varphi(q)](E)$ for all $E \in B(\mathbb{R})$ and all $q \in Q(X)$. To see that ψ is well defined, suppose $q_1(E) = q_2(F)$ for q_1, q_2 in $Q(X)$, E, F in $B(\mathbb{R})$. Then $\chi_E \cdot q_1 = \chi_F \cdot q_2$, so $\chi_E \cdot \varphi(q_1) = \varphi(\chi_E \cdot q_1) = \varphi(\chi_F \cdot q_2) = \chi_F \cdot \varphi(q_2)$. Thus $\varphi(q_1)(E) = \varphi(q_2)(F)$, showing ψ is well defined. By construction, for each $q \in Q(X)$, $\psi \circ q = \varphi(q) \in Q(X')$. By the induced map property, $\psi \in \hom[X, X']$ and $Q(\psi) = \varphi$, so Q is surjective, completing the proof. □

We will now give some partial characterizations of the induced map property. We will characterize those $X \in \mathscr{X}$ such that (X, X') has the induced map property for all X', and the analogous result with the order reversed. For the former, the key condition is weak separability.

> *Definition.* A system of measurements $(X, \mathscr{M})$ is *weakly separable* if for each $M \in \mathscr{M}$ there exists a pair (B, M_1) with $M \subseteq B \subseteq M_1$, $M_1 \in \mathscr{M}$ and B a separable Boolean sub-σ-algebra of M_1.

If $(X, \mathscr{M})$ is a system of measurements, then $\mathscr{S}(\mathscr{M})$ will denote the collection of separable Boolean sub-σ-algebras of members of $\mathscr{M}$. Note that $(X, \mathscr{S}(\mathscr{M}))$ is weakly separable.

> *Theorem 4.2.* If $(X, \mathscr{M})$ is a system of measurements, then (X, X') has the induced map property for all $X' \in \mathscr{X}$ iff X is weakly separable.

Proof. Assume that X is weakly separable. To verify that (X, X') has the induced map property for all $X' \in \mathscr{X}$, assume ψ: $X \to X'$ and $\psi \circ q \in Q(X')$ for all $q \in Q(X)$. Let M be in $\mathscr{M}$; by weak separability we can choose (B, M_1) with $M \subseteq B \subseteq M_1, M_1 \in \mathscr{M}, B$ separable. Since $B(\mathbb{R})$ is the free Boolean σ-algebra on countably many generators (cf. Ramsay, 1966) there exists an observable $q \in Q(X)$ with range B. By hypothesis $\psi \circ q \in Q(X')$; in particular, there exists $M' \in \mathscr{M}'$ such that $\psi \circ q$ is a σ-homomorphism of $B(\mathbb{R})$ into M'. Thus $\psi(M) \subseteq \psi(B) = \psi[q(B(\mathbb{R}))] \subseteq M'$. To see that $\psi|_M$ is a σ-homomorphism, suppose $\{x_i\}$ is a sequence in M. Then let $\{E_i\}$ in $B(\mathbb{R})$ be chosen so $q(E_i) = x_i$; we then have

$$\psi(\vee x_i) = \psi(q(\cup E_i)) = \vee(\psi \circ q)(E_i) = \vee \psi(x_i)$$

and similarly $\psi(x') = \psi(x)'$, so ψ is a homomorphism, establishing that (X, X') has the induced map property.

Conversely, assume (X, X') has the induced map property for all X'. Let X' be the system of measurements $(X, \mathscr{S}(\mathscr{M}))$. Let ψ be the identity map on X and let $q \in Q(X, \mathscr{M})$. The range of q will be separable, so $q(B(\mathbb{R})) \in \mathscr{S}(\mathscr{M})$; therefore, $\psi \circ q = q$ is in $Q(X, \mathscr{S}(\mathscr{M}))$. By the induced map property ψ is a homomorphism from $(X, \mathscr{M})$ to $(X, \mathscr{S}(\mathscr{M}))$. It follows that for each $M \in \mathscr{M}$ there exists $B \in \mathscr{S}(\mathscr{M})$ with $M = \psi(M) \subseteq B$, which shows that $(X, \mathscr{M})$ is weakly separable. □

> *Corollary 4.3.* The functor Q is an equivalence from the category of weakly separable systems of measurements to the category of standard systems of observables.

Proof. Let $\mathscr{X}_{\mathrm{WS}}$ be the category of weakly separable systems of measurements. By virtue of Theorems 4.1 and 4.2, Q: $\hom[X, X'] \to \hom[Q(X), Q(X')]$ is bijective for all X, X' in $\mathscr{X}_{\mathrm{WS}}$. Now let Q_0 be any system in $\mathscr{Q}$. By Theorem 3.2 there exists $(X, \mathscr{M}) \in \mathscr{X}$ with $Q_0 \cong Q(X, \mathscr{M})$. But $Q(X, \mathscr{M}) = Q(X, \mathscr{S}(\mathscr{M}))$ and $(X, \mathscr{S}(\mathscr{M})) \in \mathscr{X}_{\mathrm{WS}}$, so (4.1) holds and the functor Q is an equivalence from $\mathscr{X}_{\mathrm{WS}}$ to $\mathscr{Q}$. □

It follows from Corollary 4.3 that a system $X \in \mathscr{X}_{WS}$ is determined up to isomorphism by its associated system of observables $Q(X)$. An explicit correspondence is given by the construction in Lemma 4.1. If Q_0 is any standard system of observables, then $(X(Q_0), \mathscr{M}(Q_0))$ is weakly separable and $Q_0 \cong Q(X(Q_0), \mathscr{M}(Q_0))$. Conversely, if X_0 is weakly separable, then $X_0 \cong (X[Q(X_0)], \mathscr{M}[Q(X_0)])$. [See Shultz (1972, Theorem 3.39) for proofs of these results, which will not be used in the sequel.]

We next investigate those systems X such that (X', X) has the induced map property for all $X' \in \mathscr{X}$. The key property is that of countable compatibility.

Definition. A system of measurements X is said to have the *countable compatibility* property if whenever $A \subseteq X$ is such that every countable subset of A is compatible, then A is compatible.

Lemma 4.4. Let $(X, \mathscr{M})$ be a system of measurements, and let $\mathscr{C}(\mathscr{M})$ be the collection of subsets A of X satisfying

every countable subset of A is contained in some $M \in \mathscr{M}$; (4.3)

A is closed under complements and countable joins. (4.4)

If each $A \in \mathscr{C}(\mathscr{M})$ is endowed with the Boolean σ-algebra structure inherited from the collection $\mathscr{M}$, then $(X, \mathscr{C}(\mathscr{M}))$ is a system of measurements satisfying the countable compatibility property.

Proof. We omit the straightforward verification that $(X, \mathscr{C}(\mathscr{M}))$ is a system of measurements. Now suppose $B \subseteq X$ is such that every countable subset of B is compatible [i.e., contained in some $A \in \mathscr{C}(\mathscr{M})$]. Then, clearly, by construction of $\mathscr{C}(\mathscr{M})$ every countable subset of B is also contained in some $M \in \mathscr{M}$.

For each countable subset $E \subseteq B$ let $[E]$ denote the Boolean sub-σ-algebra generated by E in any $M \in \mathscr{M}$ containing E. (Note that $[E]$ is independent of the choice of M.) Now define $\tilde{B}$ to be the union of the sets $\{[E] \mid E \text{ is a countable subset of } B\}$. We claim $\tilde{B}$ is in $\mathscr{C}(\mathscr{M})$. For if $\{b_i\}$ is a countable subset of $\tilde{B}$, then there exist countable subsets $E_i \subseteq B$ with $b_i \in [E_i]$ for each i. Thus $\{b_i\} \subseteq [\cup E_i]$ and the latter is contained in some $M \in \mathscr{M}$, and in fact is a Boolean sub-σ-algebra of M. Note $\vee b_i$ and each b_i' are also in $[\cup E_i] \subseteq \tilde{B}$, and thus (4.3) and (4.4) hold, i.e., $\tilde{B} \in \mathscr{C}(\mathscr{M})$. Thus B is compatible in $(X, \mathscr{C}(\mathscr{M}))$, and we have shown the countable compatibility property holds for $(X, \mathscr{C}(\mathscr{M}))$. □

Theorem 4.5. Let $(X, \mathscr{M})$ be a system of measurements. Then (X', X) has the induced map property for all $X' \in \mathscr{X}$ iff X satisfies the countable compatibility property.

Proof. Suppose first that (X', X) has the induced map property for all $X' \in X$. Then, in particular, the pair $([X, \mathscr{C}(\mathscr{M})], (X, \mathscr{M}))$ has the induced map property. Let ψ be the identity map on X, and let $q \in Q(X, \mathscr{C}(\mathscr{M}))$. Note that if $A \in \mathscr{C}(\mathscr{M})$, then all Boolean sub-$\sigma$-algebras of A are in $\mathscr{C}(\mathscr{M})$;

in particular, the range $q(B(\mathbb{R}))$ of q is in $\mathscr{C}(\mathscr{M})$. Choose a sequence $\{x_i\}$ in the range of q which generates the range. By definition of $\mathscr{C}(\mathscr{M})$, there exists $M \in \mathscr{M}$ containing the sequence $\{x_i\}$. Now $M \cap q(B(\mathbb{R}))$ is a Boolean sub-σ-algebra of the range containing all $\{x_i\}$, so it coincides with the range; in particular, $q(B(\mathbb{R})) \subseteq M$. Thus $q = \psi \circ q$ is in $Q(X,\mathscr{M})$. By the induced map property ψ is a homomorphism from $(X,\mathscr{C}(\mathscr{M}))$ to $(X,\mathscr{M})$. Now suppose $D \subseteq X$ has the property that every countable subset of D is compatible in $(X,\mathscr{M})$. Then since $\mathscr{M} \subseteq \mathscr{C}(\mathscr{M})$, the same property holds in $(X,\mathscr{C}(\mathscr{M}))$. By Lemma 4.4 it follows that D is contained in some $A \in \mathscr{C}(\mathscr{M})$. Now since the identity map ψ is a homomorphism, then $D = \psi(D) \subseteq \psi(A) \subseteq M \in \mathscr{M}$ for some $M \in \mathscr{M}$. We have thus shown $(X,\mathscr{M})$ has the countable compatibility property.

Conversely, assume $(X,\mathscr{M})$ has the countable compatibility property. Given $(X',\mathscr{M}') \in \mathscr{X}$ and $\psi: X' \to X$ assume $\psi \circ q \in Q(X)$ for all $q \in Q(X')$. Given $M' \in \mathscr{M}'$ we would like to show that there exists $M \in \mathscr{M}$ with $\psi(M') \subseteq M$, i.e., that $\psi(M')$ is compatible. By hypothesis, it suffices to show every countable subset of $\psi(M')$ is compatible. Thus let $\{y_i\}$ be a given sequence in $\psi(M')$ and choose $\{x_i\} \subseteq M'$ such that $\psi(x_i) = y_i$ for all i. Now choose an observable $q \in Q(X',\mathscr{M}')$ such that the sequence $\{x_i\}$ is in the range of q [possible since $B(\mathbb{R})$ is free on countably many generators]. Then $\psi \circ q \in Q(X)$ implies that there exists $M \in \mathscr{M}$ such that $\psi[q(B(\mathbb{R}))] \subseteq M$; in particular, $\psi(x_i) \in M$ for all i, which shows that $\psi(M')$ is compatible. As in the proof of Theorem 4.2, $\psi \circ q \in Q(X)$ for all $q \in Q'$ implies that ψ preserves complements and countable joins, so $\psi|_{M'}$ is a σ-homomorphism from M' to M. Thus ψ is a homomorphism, and the proof that (X', X) has the induced map property is complete. □

Corollary 4.6. Q is an equivalence from the category of systems of measurements with the countable compatibility property to the category of standard systems of observables.

Proof. Let $\mathscr{X}_{CC}$ be the category of systems of measurements with the countable compatibility property. Then, by Theorems 4.1, 4.5, Q: hom$[X, X'] \to$ hom$[Q(X), Q(X')]$ is bijective for all X, X' in $\mathscr{X}_{CC}$. Now suppose $Q_0 \in \mathscr{Q}$; then, by Theorem 3.2, there exists $(X,\mathscr{M}) \in \mathscr{X}$ such that $Q(X,\mathscr{M}) \cong Q_0$. By Lemma 4.4, $(X,\mathscr{C}(\mathscr{M}))$ is in $\mathscr{X}_{CC}$, and in the proof of Theorem 4.5 it was shown that $Q(X,\mathscr{M}) = Q(X,\mathscr{C}(\mathscr{M}))$; this completes the proof that Q: $\mathscr{X}_{CC} \to \mathscr{Q}$ is an equivalence. □

Corollary 4.7. Quantum logics L_1 and L_2 are isomorphic iff the associated systems of observables $Q(L_1)$ and $Q(L_2)$ are isomorphic.

Proof. As previously observed, $h: L_1 \to L_2$ is a homomorphism iff $h: (L_1,\mathscr{M}(L_1)) \to (L_2,\mathscr{M}(L_2))$ is a homomorphism. By Theorem 2.2 $(L,\mathscr{M}(L))$ satisfies the countable compatibility property for every quantum logic L. Thus by Corollary 4.6, Q: hom$[L_1, L_2] \to$ hom$[Q(L_1), Q(L_2)]$ is bijective and the statement in Corollary 4.7 follows. □

As with $\mathscr{X}_{\mathrm{WS}}$, we can describe explicitly the construction that is the inverse of the functor Q: $\mathscr{X}_{\mathrm{CC}} \to \mathscr{Q}$. If Q_0 is a standard system of observables then $(X(Q_0), \mathscr{C}[\mathscr{M}(Q_0)])$ is a system of measurements with the countable compatibility property, and $Q_0 \cong Q(X(Q_0), \mathscr{C}[\mathscr{M}(Q_0)])$. If X_0 is any countably compatible system of measurements, then $X_0 \cong (X[Q(X_0)], \mathscr{C}\{\mathscr{M}[Q(X_0)]\})$. [See Shultz (1972, Theorem 3.45) for proofs.]

It is natural to inquire if there is a subcategory $\mathscr{X}_0$ of $\mathscr{X}$ that would contain $\mathscr{X}_{\mathrm{WS}}$ and $\mathscr{X}_{\mathrm{CC}}$ and for which Q: $\mathscr{X}_0 \to \mathscr{Q}$ would be an equivalence. However, this is not the case, as shown by the following example: Let B be a Boolean σ-algebra that is not separable. Let $X_1 = (B, \{B\})$ and $X_2 = (B, \mathscr{S}(B))$. Note that $X_1 \in \mathscr{X}_{\mathrm{CC}}$ and $X_2 \in \mathscr{X}_{\mathrm{WS}}$. We claim the induced map property fails for the pair (X_1, X_2). Let ψ be the identity map on B. Since the range of every observable $q \in Q(X_1)$ is separable, then $q = \psi \circ q \in Q(X_2)$. But ψ: $X_1 \to X_2$ is not a homomorphism since $\psi(B) = B$ is not contained in any $M \in \mathscr{S}(B)$. Thus (X_1, X_2) does not have the induced map property. Now if $\mathscr{X}_{\mathrm{WS}} \subseteq \mathscr{X}_0$ and $\mathscr{X}_{\mathrm{CC}} \subseteq \mathscr{X}_0$, then X_1 and X_2 are in $\mathscr{X}_0$ so Q: $\mathscr{X}_0 \to \mathscr{Q}$ is not an equivalence by Theorem 4.1.

As we have seen, the lack of equivalence of $\mathscr{X}$ and $\mathscr{Q}$ is reflected in the fact that in passing from X to $Q(X)$ information is sometimes lost. For example, $(X, \mathscr{M})$, $(X, \mathscr{S}(\mathscr{M}))$, and $(X, \mathscr{C}(\mathscr{M}))$ are in general not isomorphic, yet they have the same associated system of observables. Note that the three systems differ with regard to compatibility of uncountable sets only; one could argue that physically such systems cannot be distinguished.

Note that if $X_1, X_2 \in \mathscr{X}$, then ψ: $X_1 \to X_2$ is a homomorphism iff ψ preserves (i) compatibility and (ii) complements and countable joins on compatible subsets. We propose to weaken (i).

> *Definition.* Let $(X, \mathscr{M})$ and $(X', \mathscr{M}')$ be systems of measurements. Then ψ: $X \to X'$ is a *quasihomomorphism* if ψ preserves (i) compatibility of countable subsets of X, and (ii) complements and countable joins on compatible subsets of X.

It is perhaps natural now to define a quasiobservable of $(X, \mathscr{M})$ as a quasihomomorphism from $(B(\mathbb{R}), \{B(\mathbb{R})\})$ into $(X, \mathscr{M})$. However, it is not difficult to verify that this concept is redundant: that quasiobservables and observables coincide.

Let $\tilde{\mathscr{X}}$ be the category of systems of measurements with quasihomomorphisms as morphisms. Define the functor Q: $\tilde{\mathscr{X}} \to \mathscr{Q}$ as before, i.e., Q takes $X \in \tilde{\mathscr{X}}$ to $Q(X)$ and $\psi \in \hom[X, X']$ to the map $q \to \psi \circ q$ in $\hom[Q(X), Q(X')]$. We now show Q: $\tilde{\mathscr{X}} \to \mathscr{Q}$ is an equivalence.

> *Theorem 4.8.* Q is an equivalence from the category of systems of measurements and quasihomomorphisms to the category of standard systems of observables.

Proof. By Theorem 3.2, for each $Q_0 \in \mathscr{Q}$ there exists $X \in \tilde{\mathscr{X}}$ with $Q(X) \cong Q_0$. The argument in the proof of Theorem 4.1 applies here also to show Q: $\hom[X, X'] \to \hom[Q(X), Q(X')]$ is one to one for all X, X' in $\tilde{\mathscr{X}}$.

Following that same proof, suppose $\varphi \in \hom[Q(X), Q(X')]$ and define $\psi: X \to X'$ by $\psi(q(E)) = [\varphi(q)](E)$ for all $E \in B(\mathbb{R})$, $q \in Q(X)$. For each compatible sequence $\{x_i\}$ in X choose $\{E_i\}$ in $B(\mathbb{R})$ with $q(E_i) = x_i$ for all i; since $\psi \circ q = \varphi(q) \in Q(X')$, then $\{\psi(q(E_i))\} = \{\psi(x_i)\}$ are contained in some $M \in \mathcal{M}'$; thus ψ preserves compatibility of countable subsets of X. Since $\psi(\vee x_i) = (\psi \circ q)(\vee E_i) = \vee(\psi \circ q)(E_i) = \vee \psi(x_i)$, then ψ preserves countable joins of compatible sets and similarly ψ preserves complements. Thus ψ is a quasihomomorphism from X to X'. Since $Q(\psi) = \varphi$, we have completed the proof that Q is an equivalence. □

In summary, to achieve equivalence of the event and observable frameworks, we can either restrict the category of event systems (e.g., to $\mathscr{X}_{\mathrm{WS}}$ or $\mathscr{X}_{\mathrm{CC}}$), or weaken the notion of homomorphism.

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